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Endo-circulant digraphs: connectivity and generalized cycles[1]

J.M. Brunat, M. Maureso *, M. Mora

Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, C. Pau Gargallo, 5. E-08028 Barcelona, Spain

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Abstract

Given a finite abelian group A, a subset $\Delta \subset A$ and an endomorphism ϕ of A, the endo-circulant digraph $G_A(\phi, \Delta)$ is defined by taking A as vertex set and every vertex x adjacent to the vertices of the form $\phi(x) + a$ with $a \in \Delta$. In this paper, the endo-circulant digraphs which are strongly connected are characterized and so are those that are generalized cycles. Moreover, a sufficient condition is obtained for an endo-circulant digraph to be a Cayley digraph. © 1999 Elsevier Science B.V. All rights reserved

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1. Introduction

Some families of digraphs with good diameter, routing or connectivity, proposed as a model for interconnection networks, are c-circulant digraphs. This class of digraphs was defined in [8,9] as follows: Let N be a positive integer, Δ a subset of $\mathbf{Z}_N$, and $c \in \mathbf{Z}$. The *c-circulant digraph* $G_N(c, \Delta)$ has $\mathbf{Z}_N$ as a set of vertices and adjacency rules given by $x \to cx + a$ with $a \in \Delta$. If $c = 1$, we have a *circulant digraph*, i.e., a Cayley digraph on the cyclic group $\mathbf{Z}_N$. The c-circulant digraphs with the set Δ of the form $\Delta = \{a, a+1, \ldots, a+d-1\}$ are called *consecutive-d digraphs* and have been widely studied by Cao et al., see [2] and its references.

In this paper, we study a natural generalization of c-circulant digraphs, the endo(morphism)-circulant digraphs. Let A be a finite abelian group, ϕ an endomorphism of A and Δ a subset of A. The *endo-circulant digraph* $G_A(\phi, \Delta)$ is the digraph that has the elements of A as vertices and the pairs $(x, \phi(x) + a)$ with $x \in A$ and $a \in \Delta$

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* Corresponding author. E-mail: maureso@grec.ma2.upc.es.

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as arcs. In particular, the c-circulant digraphs are obtained by taking $A = \boldsymbol{Z}_N$ and ϕ as the endomorphism of $\boldsymbol{Z}_N$ defined by $\phi(x) = cx$. Moreover, the endo-circulant digraphs generalize the Cayley digraphs on finite abelian groups, or multidimensional circulant (see [4,7]), which are obtained when the endomorphism ϕ is the identity mapping of the group.

Examples of c-circulant digraphs are the De Bruijn digraphs $B(d,k)$. Recall that the vertices of $B(d,k)$ are the sequences $x_1 \cdots x_k$ with $x_i \in \boldsymbol{Z}_d$ and each vertex $x_1 \cdots x_k$ is adjacent to the d vertices $x_2 \cdots x_k x$ with $x \in \boldsymbol{Z}_d$. It is known that $B(d,k)$ is isomorphic to the c-circulant digraph $B(d,k) = G_{d^k}(d, \{0,1,\dots,d-1\})$ (see [12]). Nevertheless, the above definition of $B(d,k)$ fits better with the endo-circulant digraph $G_A(\phi, \Delta)$, where $A = \boldsymbol{Z}_d^k$, the endomorphism ϕ is the *perfect shuffle* defined by $\phi(x_1,\dots,x_k) = (x_2,\dots,x_k,x_1)$, and $\Delta = \{(0,\dots,0,x) \colon x \in \boldsymbol{Z}_d\}$.

The digraphs $C_n(d,k)$, studied in [11] in the context of high arc transitivity and in [7] as a model of dynamic memory networks, provide other examples of endo-circulant digraphs. The vertices of $C_n(d,k)$ are the elements of $\boldsymbol{Z}_n \times \boldsymbol{Z}_d^k$ and each vertex $(i; x_1,\dots,x_k)$ is adjacent to the vertices $(i+1; x_2,\dots,x_k,x)$ with $x \in \boldsymbol{Z}_d$. If we take $A = \boldsymbol{Z}_n \times \boldsymbol{Z}_d^k$, the endomorphism ϕ defined by $\phi(i; x_1,\dots,x_k) = (i; x_2,\dots,x_k,x_1)$ and $\Delta = \{(1; 0,\dots,0,x) \colon x \in \boldsymbol{Z}_d\}$ we get $C_n(d,k) = G_A(\phi, \Delta)$.

In [1,8,9,14], some problems for c-circulant digraphs have been studied, such as characterizing those that are (strongly) connected, characterizing those that are generalized cycles, deciding if the line digraph preserves the property of being c-circulant, properties related to the isomorphism problem and others. We believe that many of these properties can be generalized to endo-circulant digraphs. In this paper, we focus on two problems: the characterization of the endo-circulant digraphs that are connected (Section 2), and the characterization of those that are generalized cycles (Section 3). Moreover, by using the structure of generalized cycles, we give a sufficient condition for an endo-circulant digraph to be a Cayley digraph.

For non-defined graph-theoretical concepts we refer to [3]. Throughout this paper, A denotes a finite abelian group and I the identity mapping of A.

2. Connectivity

Let x, y be the vertices of a digraph G. A *walk* of length l is a sequence of vertices $x = x_0 x_1 \cdots x_l = y$ such that (x_{i-1}, x_i) is an arc for $1 \leqslant i \leqslant l$. A *path* is a walk without vertex repetition. It is known that the existence of a walk from x to y is equivalent to the existence of a path from x to y (see [3]). A digraph G is (strongly) *connected* if there exists a path between each pair of vertices. In this section, we shall characterize the endo-circulant digraphs that are connected. First, we need some algebraic lemmas.

Let ϕ be an endomorphism of a group A. For $i \geqslant 1$, define the endomorphisms $s_i(\phi)$ of A by

$$s_i(\phi) = I + \phi + \phi^2 + \cdots + \phi^{i-1}$$

and take $s_0(\phi) = 0$. The properties of the following lemma are easily checked.

Lemma 1. *Let ϕ be an endomorphism of a group and i, j non-negative integers. Then*
(i) $\phi^i - I = (\phi - I)s_i(\phi)$;
(ii) $s_{i+j}(\phi) - s_i(\phi) = \phi^i s_j(\phi)$;
(iii) $s_{ij}(\phi) = s_i(\phi)s_j(\phi^i)$.

Lemma 2. *Let ϕ be an automorphism of A and $n \geqslant 1$ an integer. Then, there is an integer $k \geqslant n$ such that $\phi^k = I$ and $s_k(\phi) = 0$.*

Proof. Let t_1 be the order of ϕ in the group of automorphisms of A, and let t_2 be the order of $s_{t_1}(\phi)$ in the additive group of the endomorphisms of A. If $t = t_1 t_2$, we have $\phi^t = I$ and

$$s_t(\phi) = s_{t_1}(\phi)s_{t_2}(\phi^{t_1}) = s_{t_1}(\phi)s_{t_2}(I) = t_2 s_{t_1}(\phi) = 0.$$

For the given n, take l such that $lt > n$ and $k = lt$. Then $\phi^k = I$ and

$$s_k(\phi) = s_t(\phi)s_l(\phi^t) = 0. \qquad \square$$

Lemma 3. *Let H be a subgroup of A of index m, ϕ an automorphism of A such that $\phi(H) = H$, and $a \in A$. Then the set $\{s_i(\phi)(a)\colon 0 \leqslant i\}$ contains a system of representatives of A/H if and only if the set $\{s_i(\phi)(a)\colon 0 \leqslant i \leqslant m-1\}$ is a system of representatives of A/H.*

Proof. The sufficiency is obvious. Assume that the set $\{s_i(\phi)(a)\colon 0 \leqslant i\}$ contains a system of representatives of A/H. We shall show that every pair of elements of $\{s_i(\phi)(a)\colon 0 \leqslant i \leqslant m-1\}$ are not equivalent modulo H. Suppose that $s_{i+j}(\phi)(a) - s_i(\phi)(a) \in H$ with $0 \leqslant i < i+j \leqslant m-1$. Then, by using Lemma 1 and $\phi(H) = H$, we have $s_j(\phi)(a) \in H$. Therefore, for any $t \geqslant 0$

$$s_{j+t}(\phi)(a) - s_t(\phi)(a) = \phi^t s_j(\phi)(a) \in H.$$

Then the set $\{s_t(\phi)(a)\colon 0 \leqslant t < j\}$ has cardinality $< m$ and contains a system of representatives of A/H, which is a contradiction. $\square$

If ϕ is an endomorphism of A and $S \subset A$, the smallest subgroup H of A such that $S \subset H$ and $\phi(H) \subset H$ will be denoted by $\langle \phi, S \rangle$. It is the subgroup generated by the elements of the form $\phi^k(s)$ with $s \in S$ and $k \geqslant 0$.

Let $a \in \Delta \subset A$ and define $\Delta(a) = \{b - a\colon b \in \Delta,\ b \neq a\}$. The *difference* subgroup of $G_A(\phi, \Delta)$ is the group $\mathscr{D} = \langle \phi, \Delta(a) \rangle$. It is independent of the particular $a \in \Delta$ selected, because the subgroup generated by $\Delta(a)$ coincides with the subgroup generated by all the differences $b_1 - b_2$, with $b_1, b_2 \in \Delta$.

To study the connectivity, let us first consider the case when ϕ is an automorphism.

Proposition 4. *Let $G = G_A(\phi, \Delta)$ be an endo-circulant digraph, $\mathscr{D}$ the difference subgroup of G, m the index of $\mathscr{D}$ in A and $a \in \Delta$. If ϕ is an automorphism of A, then G is* *connected if and only if the set*

$$\{s_i(\phi)(a)\colon\ 0 \leqslant i \leqslant m-1\}$$

is a system of representatives of $A/\mathscr{D}$.

Proof. Since ϕ is an automorphism, the digraph G is $|\Delta|$-regular. From the fact that every regular graph of even degree is 2-factorable [10], it follows that every regular digraph admits a 1-factorization. Hence every arc belongs to a cycle. Therefore, the existence of a path from a vertex x to a vertex y is equivalent to the existence of a walk from y to x. It follows that the digraph is connected if and only if there is a walk from a fixed vertex to any vertex. Thus, the endo-circulant digraph G is connected if and only if there is a path from 0 to x for all $x \in A$.

Assume that G is connected. Let x be a vertex of G and consider a path from 0 to x, say $0 = x_0 x_1 \dots x_{l-1} x_l = x$. If $x_i = \phi(x_{i-1}) + a_i$, $1 \leqslant i \leqslant l$, we have

$$x = \phi^{l-1}(a_1) + \phi^{l-2}(a_2) + \cdots + a_l$$

$$= \sum_{i=0}^{l-1} \phi^i(a_{l-i} - a) + \sum_{i=0}^{l-1} \phi^i(a)$$

$$= \sum_{i=0}^{l-1} \phi^i(a_{l-i} - a) + s_l(\phi)(a).$$

Hence, $x - s_l(\phi)(a) \in \mathscr{D}$ and the set $\{s_i(\phi)(a)\colon\ 0 \leqslant i\}$ contains a system of representatives of $A/\mathscr{D}$. By applying Lemma 3, we obtain that $\{s_i(\phi)(a)\colon\ 0 \leqslant i \leqslant m-1\}$ is a system of representatives of $A/\mathscr{D}$.

Conversely, if x is an element of A, then there exists an integer $l \geqslant 0$ and a $y \in \mathscr{D}$ such that $x = s_l(\phi)(a) + y$. Let $t = \min\{k \geqslant 1\colon\ \phi^k = I\}$. By definition of $\mathscr{D}$, we can write y in the form

$$y = \sum_{j=1}^{n} \lambda_j \phi^{r_j}(a_j - a),$$

where n, λ_j and r_j are integers, $0 \leqslant r_1 \leqslant r_2 \leqslant \cdots \leqslant r_n \leqslant t-1, 1 \leqslant \lambda_j \leqslant |\mathscr{D}|$ and $a_j \in \Delta$, for $1 \leqslant j \leqslant n$. Each term of the above sum can be expressed in the form

$$\lambda_j \phi^{r_j}(a_j - a) = (\phi^{r_j + tp_j} + \phi^{r_j + t(p_j+1)} + \cdots + \phi^{r_j + t(p_{j+1}-1)})(a_j - a)$$

with $p_1 = 0$ and $p_j = \lambda_1 + \cdots + \lambda_{j-1}$ for $2 \leqslant j \leqslant n+1$. By applying Lemma 2, there exists $q > r_n + tp_{n+1}$ such that $\phi^q = I$ and $s_q(\phi) = 0$. From Lemma 1, it follows that $s_{l+q}(\phi)(a) = s_l(\phi)(a)$. Therefore,

$$x = s_{q+l}(\phi)(a) + y$$

$$= \sum_{j=0}^{q+l-1} \phi^j(a) + \sum_{j=1}^{n} \sum_{i=p_j}^{p_{j+1}-1} \phi^{r_i+ti}(a_j - a)$$

$$= \sum_{j=0}^{q+l-1} \phi^j(b_j)$$

with $b_j \in \Delta$. Thus, there exists a walk from 0 to x. □

The above proposition can easily be used to prove that the digraphs $C_n(d,k)$ are connected. Indeed, we have $\mathscr{D} = \{0\} \times \boldsymbol{Z}_d^k$ and the index of $\mathscr{D}$ in A is n. If we take $a = (1; 0, \ldots, 0) \in \Delta$, we have $s_i(\phi)(a) = (i; 0, \ldots, 0)$, for all $i \in \boldsymbol{Z}_n$. So the condition of Proposition 4 is satisfied and the digraphs $C_n(d,k)$ are connected.

Next, we study the case when ϕ is not necessarilyan automorphism. Let $G = G_A(\phi, \Delta)$ be an endo-circulant digraph. The endo-circulant digraph defined by the subgroup $\phi(A)$ of A, the restriction of ϕ to $\phi(A)$ (which will be also denoted by ϕ) and the subset $\phi(\Delta)$ of $\phi(A)$ will be denoted by ϕG. For $i \geqslant 2$, the digraph $\phi^i G$ is iteratively defined by $\phi^i G = \phi(\phi^{i-1} G)$.

Recall that if $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are two digraphs, a *graph homomorphism* from G_1 to G_2 is a mapping $f : V_1 \to V_2$ such that if $(x, y) \in E_1$ then $(f(x), f(y)) \in E_2$. Note that, if $G = G_A(\phi, \Delta)$ is an endo-circulant digraph, the mapping $\phi : A \to \phi(A)$ is a group homomorphism and also a digraph exhaustive homomorphism from G to ϕG.

The following proposition shows that the problem of deciding if $G = G_A(\phi, \Delta)$ is connected can be reduced to the study of the connectivity of the digraph ϕG.

Proposition 5. *An endo-circulant digraph $G = G_A(\phi, \Delta)$ is connected if and only if the following conditions hold*:

(i) *the set Δ contains a system of representatives of $A/\phi(A)$*;

(ii) *the digraph ϕG is connected.*

Proof. Suppose that G is connected.

(i) Given $x \in A$, then there is an element $y \in A$ adjacent to x. For some $a \in \Delta$, we have $x = \phi(y) + a$. Hence a is a representative of the coset $x + \phi(A)$.

(ii) Let $\phi(x)$ and $\phi(y)$ be two vertices of $\phi(A)$. Since G is connected, there exists a path from x to y in G. As ϕ is a graph homomorphism, by applying ϕ to the vertices of the path, we obtain a walk from $\phi(x)$ to $\phi(y)$ in ϕG.

Conversely, assume that conditions (i) and (ii) hold and let x, y be vertices in A. Condition (i) implies that for some $z \in A$ and some $a \in \Delta$, we have $y = \phi(z) + a$. Moreover, condition (ii) implies that there exists a path from $\phi(x)$ to $\phi(z)$ in ϕG, say

$$\phi(z) = \phi^l(\phi(x)) + \phi^{l-1}(\phi(a_1)) + \cdots + \phi(\phi(a_{l-1})) + \phi(a_l)$$

with $a_i \in \Delta$. Define $x_0 = x$, $a_{l+1} = a$ and, for $1 \leqslant i \leqslant l+1$, $x_i = \phi(x_{i-1}) + a_i$. Therefore, $x = x_0 x_1 \dots x_{l+1} = y$ is a walk in G from x to y. □

Since A is a finite group, for any endomorphism ϕ of A the descending sequence

$$A \supset \phi(A) \supset \phi^2(A) \supset \phi^3(A) \supset \cdots$$

becomes equality for some iteration. Denote by $r = r(\phi)$ the minimum integer r such that $\phi^r(A) = \phi^{r+1}(A)$. The restriction of ϕ to $\phi^r(A)$ is an automorphism of $\phi^r(A)$. In fact, $\phi^r(A)$ is the largest subgroup of A such that the restriction of ϕ to it is an automorphism.

Proposition 6. *Let $G = G_A(\phi, \Delta)$ be an endo-circulant digraph and $r = r(\phi)$. Then G is connected if and only if the following conditions hold*:

(i) *the set Δ contains a system of representatives of $A/\phi(A)$*;

(ii) *the digraph $\phi^r G$ is connected.*

Proof. It is easily checked that if Δ contains a system of representatives of $A/\phi(A)$, then $\phi(\Delta)$ contains a system of representatives of $\phi(A)/\phi^2(A)$. Then, by Proposition 5, G is connected if and only if condition (i) holds and $\phi^r G$ is connected. □

As an example, take the group $A = \boldsymbol{Z}_3 \times \boldsymbol{Z}_3 \times \boldsymbol{Z}_6$, the endomorphism ϕ defined by $\phi(x, y, z) = (z \bmod 3, x, 3z)$, and $\Delta = \{(2,0,0),(0,0,1),(0,0,2)\}$. It is easy to check that $\phi(A) = \langle (0,1,0),(1,0,3) \rangle$ and that Δ is a system of representatives of $A/\phi(A)$. We have $r = 3$ and $\phi^3(A) = \langle (0,0,3) \rangle$. As $\phi^3(\Delta) = \{(0,0,0),(0,0,3)\}$, the digraph $\phi^3 G_A(\phi, \Delta)$ is the complete digraph of two vertices. Therefore, Proposition 6 ensures that $G_A(\phi, \Delta)$ is connected.

Remark that for $i \geqslant 0$, the difference subgroup of $\phi^i G$ is $\phi^i(\mathscr{D})$, where $\mathscr{D}$ is the difference subgroup of G. Since the restriction of ϕ to $\phi^r(A)$ is an automorphism, by applying Proposition 4 we see that Proposition 6 can be stated in the following alternative way:

Theorem 7. *Let $G = G_A(\phi, \Delta)$ be an endo-circulant digraph, $r = r(\phi)$ and m the index of $\phi^r(\mathscr{D})$ in $\phi^r(A)$. Then G is connected if and only if the following conditions hold*:

(i) *the set Δ contains a system of representatives of $A/\phi(A)$*;

(ii) *the set $\{s_i(\phi)(\phi^r(a)) \colon 0 \leqslant i \leqslant m-1\}$ is a system of representatives of $\phi^r(A)/\phi^r(\mathscr{D})$.*

As a corollary we obtain the known characterization of connected c-circulant digraphs. Analogously with the notation $s_i(\phi)$, if $c \in \boldsymbol{Z}_N$, we define $s_0(c) = 0$ and $s_i(c) = 1 + c + \cdots + c^{i-1}$ for $i \geqslant 1$.

Corollary 8 (Mora et al. [9]). *Let $G = G_N(c, \Delta)$ be a c-circulant digraph with $\Delta = \{a_1, \dots, a_d\}$, $g = \gcd(N, c)$, N' be the greatest divisor of N such that* $\gcd(N', c) = 1$

and $m = \gcd(N', a_2 - a_1, \ldots, a_d - a_1)$. *The digraph G is connected if and only if the following conditions hold*:

(a) Δ *contains all congruence classes modulo* g;

(b) $\gcd(N', a_1, \ldots, a_d) = 1$;

(c) $\{s_i(c) \bmod m\colon 0 \leqslant i \leqslant m-1\} = \boldsymbol{Z}_m$.

Proof. Consider G as the endo-circulant digraph $G = G_A(\phi, \Delta)$ with $A = \boldsymbol{Z}_N$ and ϕ defined by $\phi(x) = cx$.

Since $\phi(A) = g\boldsymbol{Z}_N$, the quotient group $A/\phi(A)$ is isomorphic to $\boldsymbol{Z}_g$. Therefore, condition (i) of Theorem 7 is equivalent to condition (a).

We shall show that the condition (ii) is equivalent to both (b) and (c) together. It is known that ϕ is an automorphism of the subgroup of A of order n if and only if $\gcd(c, n) = 1$. Thus, the subgroup $\phi^r(A)$ is the subgroup of A of order N'.

The subgroup $\phi^r(\mathscr{D})$ is generated by the elements $c^{r+i}(a_j - a_1)$. As ϕ is an automorphism of $\phi^r(\mathscr{D})$, the elements $c^r(a_j - a_1)$ generate $\phi^r(\mathscr{D})$. Therefore, the index of $\phi^r(\mathscr{D})$ in $\phi^r(A)$ is

$$\gcd(N', c^r(a_2 - a_1), \ldots, c^r(a_d - a_1)) = \gcd(N', a_2 - a_1, \ldots, a_d - a_1) = m,$$

because $\gcd(c, N') = 1$.

The isomorphism $\boldsymbol{Z}_{N'} \to \phi^r(A)$ defined by $x \mapsto c^r x$ induces the isomorphism $\boldsymbol{Z}_m \to \phi^r(A)/\phi^r(\mathscr{D})$, defined by $x \mapsto c^r x + \phi^r(\mathscr{D})$.

Hence, if condition (ii) holds, then $\{s_i(c)a_1 \bmod m\colon 0 \leqslant i \leqslant m-1\} = \boldsymbol{Z}_m$. For some p, we have $s_p(c)a_1 \equiv 1 \bmod m$, so $\gcd(m, a_1) = 1$. Then $\{s_i(c) \bmod m\colon 0 \leqslant i \leqslant m-1\} = \boldsymbol{Z}_m$ and $\gcd(N', a_1, \ldots, a_d) = \gcd(m, a_1) = 1$. Thus, conditions (b) and (c) are satisfied. Conversely, from condition (b) we obtain $\gcd(m, a_1) = 1$ and, then, condition (c) implies that $\{s_i(c)a_1 \bmod m\colon 0 \leqslant i \leqslant m-1\} = \boldsymbol{Z}_m$. The above isomorphism gives (ii). $\square$

3. Generalized cycles

A digraph is *m-reachable* if for every pair of vertices x, y there exists a walk of exactly m arcs from x to y. A digraph is said to be *equi-reachable* if it is m-reachable for some m.

A digraph is an *m-generalized cycle* if there exists an m-partition $V_0, \ldots, V_{m-1}$ of the vertex set such that if (x, y) is an arc and $x \in V_i$, then $y \in V_{i+1}$ (where the subscripts are taken modulo m). An equivalent condition is that there exists an exhaustive digraph homomorphism from the digraph onto the directed cycle of length m. The sets V_i are called the *stable sets* of the digraph. All digraphs are 1-generalized cycles, so the concept is only interesting for $m > 1$. A digraph is said to be a *generalized cycle* if it is an m-generalized cycle for some $m > 1$. Fiol et al. [5] showed that a connected digraph is a generalized cycle if and only if it is not equi-reachable.

Our goal is to characterize the connected endo-circulant digraphs which are generalized cycles. As in the preceding section, we begin with the case when ϕ is an automorphism, and then we reduce the general case to this one via the digraphs $\phi^i G$.

Proposition 9. *Let $G = G_A(\phi, \Delta)$ be a connected endo-circulant digraph and suppose that ϕ is an automorphism of A. If m is the index of the difference subgroup $\mathscr{D}$ in A, then G is an m-generalized cycle, and the stable sets are the cosets of $\mathscr{D}$,*

$$V_i = s_i(\phi)(a) + \mathscr{D}, \quad 0 \leqslant i \leqslant m-1,$$

where $a \in \Delta$.

Proof. From Proposition 4 it follows that the set $\{s_i(\phi)(a) \colon 0 \leqslant i \leqslant m-1\}$ is a system of representatives of $A/\mathscr{D}$. Therefore $V_i = s_i(\phi)(a) + \mathscr{D}$, $0 \leqslant i \leqslant m-1$, is a partition of A. If (x, y) is an arc and $x \in V_i$, then, for some $b \in \Delta$, we have

$$\begin{aligned} y = \phi(x) + b \in \phi(s_i(\phi)(a) + \mathscr{D}) + b &= s_{i+1}(\phi)(a) - a + \phi(\mathscr{D}) + b \\ &= s_{i+1}(\phi)(a) + \mathscr{D} = V_{i+1}, \end{aligned}$$

because $\phi(\mathscr{D}) = \mathscr{D}$ and $b - a \in \mathscr{D}$. $\square$

Proposition 10. *Let $G = G_A(\phi, \Delta)$ be a connected endo-circulant digraph and m the index of the difference subgroup $\mathscr{D}$ in A. If ϕ is an automorphism of A, then G is a k-generalized cycle if and only if k divides m.*

Proof. From Proposition 9 it follows that G is an m-generalized cycle with stable sets

$$V_i = s_i(\phi)(a) + \mathscr{D}, \quad 0 \leqslant i \leqslant m-1,$$

where $a \in \Delta$.

First, suppose that k divides m. For each j, $0 \leqslant j \leqslant k-1$, let W_j be the union of the sets V_i with $i \equiv j \bmod k$. Then G is a k-generalized cycle with stable sets $W_0, \dots, W_{k-1}$.

Conversely, assume that G is a k-generalized cycle with stable sets $W_0, \dots, W_{k-1}$. It is not a restriction to assume that $0 \in W_0$.

Any $x \in \mathscr{D}$ admits an expression

$$x = \sum_{j=1}^{n} \lambda_j \phi^{r_j}(a_j - a),$$

where $a, a_j \in \Delta$ and λ_j and r_j are non-negative integers. Then, in the same way as in the proof of Proposition 4, there is an integer q large enough such that $\phi^q = I$, $s_q(\phi) = 0$ and

$$x = \sum_{j=0}^{q-1} \phi^j(b_j)$$

with $b_j \in \Delta$. Therefore, if $\Gamma_q(0)$ is the set of vertices x such that there exists a walk of length q from 0 to x, we have $\mathscr{D} \subset \Gamma_q(0)$.

Now, if $x \in \Gamma_q(0)$, the vertex x admits an expression

$$x = \sum_{j=0}^{q-1} \phi^j(a_j) = s_q(\phi)(a) + \sum_{j=0}^{q-1} \phi^j(a_j - a),$$

where $a, a_j \in \Delta$. Since $s_q(\phi)(a) = 0$, we have $x \in \mathcal{D}$ and $\Gamma_q(0) = \mathcal{D}$.

Let l be the remainder of the division of q by k. Since $0 \in W_0$, we have $V_0 = \mathcal{D} = \Gamma_q(0) \subset W_l = W_0$. This implies $V_1 \subset W_1$ and, recursively, $V_i \subset W_i$ for all i, $0 \leqslant i \leqslant m-1$. In particular, $V_0 \subset W_m$. Then, $m \equiv 0 \bmod k$. □

Now, we consider the case when ϕ is an arbitrary endomorphism.

Proposition 11. *An endo-circulant digraph $G = G_A(\phi, \Delta)$ is a k-generalized cycle if and only if ϕG is a k-generalized cycle.*

Proof. Let G be a k-generalized cycle with stable sets $W_0, \ldots, W_{k-1}$. The sets $\phi(W_0), \ldots, \phi(W_{k-1})$ form a partition of $\phi(A)$. Indeed, clearly the sets $\phi(W_i)$ are not empty and their union is $\phi(A)$. Moreover, if $\phi(W_i) \cap \phi(W_j) \neq \emptyset$, there are $x \in W_i$ and $y \in W_j$ such that $\phi(x) = \phi(y)$. By adding an $a \in \Delta$, we obtain that the vertex $z = \phi(x) + a = \phi(y) + a$ is adjacent from x and from y in G. If $z \in W_l$, then x and y belong to the stable set W_{l-1}. It follows that $W_i = W_{l-1} = W_j$ and we conclude that $\phi(W_i) = \phi(W_j)$.

Now, if $\phi(x) \in \phi(W_i)$ and $\phi(x)$ is adjacent to $\phi(y)$ in ϕG, then for some $a \in \Delta$, we have $\phi(y) = \phi^2(x) + \phi(a) = \phi(\phi(x) + a) \in \phi(W_{i-1})$. Therefore, ϕG is a k-generalized cycle.

Conversely, assume that ϕG is a k-generalized cycle and let $f : \phi G \to C_k$ be an exhaustive homomorphism from ϕG onto the directed cycle C_k of length k. The composition $f\phi : G \to C_k$ is an exhaustive homomorphism from G onto C_k. Hence G is a k-generalized cycle. □

Let $G = G_A(\phi, \Delta)$ be a connected endo-circulant digraph and $r = r(\phi)$. By applying the above proposition r times we have that G is a k-generalized cycle if and only if $\phi^r G$ is. Since the restriction of ϕ to $\phi^r(A)$ is an automorphism, Proposition 10 implies that $\phi^r G$ is a k-generalized cycle if and only if k divides the index of the difference subgroup $\phi^r(\mathcal{D})$ in $\phi^r(A)$. Therefore, we have the next characterization.

Theorem 12. *Let $G = G_A(\phi, \Delta)$ be a connected endo-circulant digraph, $r = r(\phi)$ and m the index of $\phi^r(\mathcal{D})$ in $\phi^r(A)$. Then G is a k-generalized cycle if and only if k divides m.*

Finally, we give a sufficient condition for an endo-circulant digraph to be a Cayley digraph. Recall that, given a finite group Γ and a subset $S \subset \Gamma$, the Cayley digraph $\mathrm{Cay}(\Gamma, S)$ is the digraph that has Γ as the set of vertices and the pairs (x, xs) with $x \in \Gamma$ and $s \in S$ as arcs. Sabidussi's theorem [13] characterizes Cayley graphs and can also be stated for digraphs: *a digraph G is isomorphic to a Cayley digraph if and only if its automorphism group has a subgroup that acts regularly on the set of vertices.* By using Sabidussi's theorem and the structure of generalized cycle, the authors gave in [1] a sufficient condition for a c-circulant digraph to be a Cayley digraph. The next proposition generalizes this result for endo-circulant digraphs.

Proposition 13. *Let $G=G_A(\phi,\Delta)$ be a connected endo-circulant digraph and m the index of the difference subgroup $\mathscr{D}$ in A. If $\phi^m=I$, then G is a Cayley digraph.*

Proof. If $m=1$ then $\phi=I$ and G is a Cayley digraph. From now on we suppose $m>1$.

From Proposition 9 it follows that G is an m-generalized cycle with stable sets $V_i=s_i(\phi)(a)+\mathscr{D}$, where $a\in\Delta$ and $0\leqslant i\leqslant m-1$. For $x\in A$, we denote by $e(x)$ the subscript such that $x\in V_{e(x)}$. For each $h\in A$ define the mapping $f_h:A\to A$ by

$$f_h(x)=x+\phi^{e(x)}(h).$$

We shall prove that f_h is an automorphism of the digraph G.

If $x-y\in\mathscr{D}$, then $e(x)=e(y)$ and $f_h(x)-f_h(y)=x-y\in\mathscr{D}$. Thus, f_h maps a coset of $\mathscr{D}$ to a coset of $\mathscr{D}$. Since the restriction of f_h to V_i is injective, for each V_i there exists a V_j such that $f(V_i)=V_j$. In particular, $f_h(V_0)=V_{e(h)}$. We shall show that $f_h(V_i)=V_{e(h)+i}$ by induction on i. For $i=0$ it is clear. If it holds for i and $y\in V_{i+1}$, the vertex y is of the form $y=\phi(x)+a$ for some $x\in V_i$ and $a\in\Delta$. Then

$$f_h(y)=\phi(x)+a+\phi^{i+1}(h)=\phi(x+\phi^i(h))+a\in\phi(V_{e(h)+i})+a\in V_{e(h)+i+1}.$$

Thus f_h is bijective. In addition, it is easy to check that (x,y) is an arc in G if and only if $(f_h(x),f_h(y))$ is. We conclude that for each $h\in A$, the mapping f_h is an automorphism of G.

The set $\Gamma=\{f_h\colon h\in A\}$ is closed under composition because

$$f_{h_1}f_{h_2}(x)=x+\phi^{e(x)}(h_2)+\phi^{e(x)+e(h_2)}(h_1)=f_{h_2+\phi^{e(h_2)}(h_1)}(x)$$

for all $x\in A$. Therefore, Γ is a subgroup of the group of automorphisms of G. If f_h stabilizes 0, then $0=f_h(0)=h$ and $f_h=I$. Hence Γ is semiregular. Now, $f_h(0)=h$ implies that Γ is transitive. By Sabidussi's theorem, G is isomorphic to the Cayley digraph $\mathrm{Cay}(\Gamma,\{f_a\colon a\in\Delta\})$. $\square$

To illustrate this proposition, consider the endo-circulant digraph $G_A(\phi,\Delta)=C_n(d,k)$ described in the introduction. The endomorphism ϕ, which is defined by $\phi(i;x_1,\ldots,x_k)=(i;x_2,\ldots,x_k,x_1)$, is an automorphism of order k. We have seen that $\mathscr{D}=\{0\}\times\boldsymbol{Z}_d^k$ and the index m of $\mathscr{D}$ in A is $m=n$. If k divides n, we have $\phi^n=I$ and we conclude that $C_n(d,k)$ is a Cayley digraph.

We remark that the converse of the above proposition is not true as was pointed out in [1] for c-circulant digraphs.

References

[1] J.M. Brunat, M. Maureso, M. Mora, Characterization of c-circulant digraphs of degree two which are circulant, Discrete Math. 165–166 (1–3) (1997) 125–137.

[2] F. Cao, D.Z. Du, D.F. Hsu, L. Hwang, W. Wu, Super line-connectivity of consecutive-d digraphs, Discrete Math. 183 (1–3) (1998) 27–38.

[3] G. Chartrand, L. Lesniak, Graphs & Digraphs, 3rd Ed., Chapman & Hall, London, 1996.

[4] M.A. Fiol, On Congruence in $\boldsymbol{Z}^n$ and the dimension of a multidimensional circulant, Discrete Math. 141 (1995) 123–134.

[5] M.A. Fiol, I. Alegre, J.L.A. Yebra, J. Fàbrega, Digraphs with walks of equal length between vertices, in: Y. Alavi et al. (Eds.), Graph Theory and its Applications to Algorithms and Computer Science, Wiley, New York, 1985, pp. 313–322.

[6] M.A. Fiol, J. Fàbrega, O. Serra, J.L.A. Yebra, A unified approach to the design and control of dynamic memory networks, Parallel Process. Lett. 3 (4) (1993) 445–456.

[7] F.T. Leighton, Circulants and the characterization of vertex-transitive graphs, J. Res. Natl. Bur. Stand. 88 (6) (1983) 395–402.

[8] M. Mora, Digrafs c-circulants com a model de xarxes d'interconnexió, Ph.D. Thesis, Universitat Politècnica de Catalunya, Barcelona, 1988.

[9] M. Mora, O. Serra, M.A. Fiol, General properties of c-circulant digraphs, Ars Combin. 25 C (1988) 241–252.

[10] J. Petersen, Die Theorie der regulären Graph, Acta Math. 15 (1891) 193–220.

[11] C.E. Praeger, Highly arc transitive digraphs, European J. Combin. 10 (1989) 281–292.

[12] S.M. Reddy, D.K. Pradhan, J.G. Kuhl, Directed graphs with minimum diameter and maximum node connectivity, Tech. Report, School of Engineering, Oakland University, Michigan, 1980.

[13] G. Sabidussi, On a class of fixed-point-free graphs, Proc. Amer. Math. Soc. 9 (1958) 800–804.

[14] O. Serra, M. Mora, M.A. Fiol, On c-circulant digraphs, Combinatorics' 88 (Ravello, 1988) vol. 2, Rende, Mediterranean, 1991, pp. 421–437.